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Test vectors and central L -values for $GL(2)$ (Automorphic Forms and Related Zeta Functions)

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Test vectors and central L -values for $\mathrm{GL}(2)$

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Abstract

This note is a write-up of our results on test vectors for $\mathrm{GL}(2)$ and their applications to central L -values, presented by the third author in the RIMS conference on automorphic forms (2014, Jan 20-24). This note also serves as a summary of our recent paper [2].

1 Local results

Let F be a local non-archimedean field of characteristic 0. Let $\mathfrak{o}, \mathfrak{p}, \varpi$ be the ring of integers, prime ideal and uniformizer for F . Let $G = \mathrm{GL}_2(F)$. Let L be a degree 2 extension of F (could be $F \oplus F$). Let π be an irreducible, admissible representation of G with conductor $c(\pi)$. Let Λ be a character of L^\times with conductor $c(\Lambda)$. Embed L^\times in $\mathrm{GL}_2(F)$ as a torus $T(F)$. Assume that $\Lambda|_{F^\times} = \omega_\pi$, the central character of π .

A natural question to ask is whether $\mathrm{Hom}_{T(F)}(\pi, \Lambda)$ is zero or not. In particular, we want to know if there exists a linear functional $\ell : V_\pi \rightarrow \mathbb{C}$ such that $\ell \neq 0$ and $\ell(\pi(t)v) = \Lambda(t)\ell(v)$ for all $t \in T(F)$ and $v \in V_\pi$. It is a theorem of Waldspurger [10] that

$$\dim \mathrm{Hom}_{T(F)}(\pi, \Lambda) \leq 1.$$

More precise information, in terms of epsilon factors, is given by the work of Saito [8] and Tunnell [9].

$$\dim \mathrm{Hom}_{T(F)}(\pi, \Lambda) = \frac{1 + \epsilon(1/2, \pi_L \times \Lambda)\omega_\pi(-1)}{2},$$

where π_L is the base change of π to $\mathrm{GL}_2(L)$. Let us assume that the above condition is satisfied for π and Λ and let $0 \neq \ell \in \mathrm{Hom}_{T(F)}(\pi, \Lambda)$. A vector $v \in V_\pi$ is called a *test vector* for ℓ if $\ell(v) \neq 0$. For applications, we need test vectors satisfying further conditions. For this purpose, let us define a *good test vector* for ℓ to be a test vector v for ℓ satisfying the following two conditions.

- i) We have $v \in V_\pi^K$, where K is a compact subgroup of G such that $\dim V_\pi^K = 1$.
- ii) The compact subgroup K in i) depends only on the conductors $c(\pi)$ and $c(\Lambda)$ of π and Λ .

Let us remark on the above two conditions. The first is a matter of computational convenience. In a trace formula application to central L -values, we need to compute some local integrals which simplify greatly if the test vector satisfies condition i) above. The second condition is crucial for average L -value applications. We wish to take an average over all new forms of a fixed level, which would mean that the local non-archimedean representation will have a fixed conductor but otherwise completely arbitrary. In [3], Gross and Prasad obtained the first results on good test vectors for $c(\pi) = 0$ or $c(\Lambda) = 0$ (under some further conditions). Let us define

$$K_n := \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^n & 1 + \mathfrak{p}^n \end{bmatrix} \cap \mathrm{GL}_2(\mathfrak{o}), \text{ for } n \geq 0.$$

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1 LOCAL RESULTS

Let us now state our first local result for the split case.

1.1 Theorem. *Let $L = F \oplus F$. Assume that π and Λ are unitary. Then $\text{Hom}_{T(F)}(\pi, \Lambda) \neq 0$, and for $0 \neq \ell \in \text{Hom}_{T(F)}(\pi, \Lambda)$ there exists a good test vector with*

$$K = \begin{bmatrix} 1 & \varpi^{-c(\Lambda)} \\ & 1 \end{bmatrix} K_{c(\pi)} \begin{bmatrix} 1 & -\varpi^{-c(\Lambda)} \\ & 1 \end{bmatrix}.$$

Proof. In the split case, the torus $T(F)$ is given by the diagonal matrices in G . Let the character Λ be given by $\Lambda(\text{diag}(x, y)) = \Lambda_1(x)\Lambda_2(y)$ for characters Λ_1, Λ_2 of F^\times . Assume, without loss of generality, that $c(\Lambda_1) \geq c(\Lambda_2)$. Write $\Lambda_1 = |\cdot|^{1/2-s_0}\mu_0$ for $s_0 \in \mathbb{C}$ and unitary character μ_0 of F^\times such that $\mu_0(\varpi) = 1$. The character Λ_2 is determined by the relation $\Lambda_1\Lambda_2 = \omega_\pi$. Let ψ be an additive character of F with conductor \mathfrak{o} and let π be given by its Whittaker model $\mathcal{W}(\pi, \psi)$. For $W \in \mathcal{W}(\pi, \psi)$, $s \in \mathbb{C}$, μ a unitary character of F^\times , define the zeta integral

$$Z(s, W, \mu^{-1}) := \int_{F^\times} W\left(\begin{bmatrix} x & \\ & 1 \end{bmatrix}\right) |x|^{s-1/2} \mu^{-1}(x) d^\times x,$$

and define $\ell : \mathcal{W}(\pi, \psi) \rightarrow \mathbb{C}$ by

$$\ell(W) := \frac{Z(s_0, W, \mu_0^{-1})}{L(s_0, \mu_0^{-1} \otimes \pi)}.$$

By the theory of zeta integrals, the above function is well-defined and belongs to $\text{Hom}_{T(F)}(\pi, \Lambda)$.

Let $W_0 \in V_\pi^{K_{c(\pi)}}$ such that $W_0(1) = 1$. Then, one can check that

$$\ell(\pi\left(\begin{bmatrix} 1 & \varpi^{-c(\Lambda)} \\ & 1 \end{bmatrix}\right)W_0) \neq 0.$$

This gives the theorem. See [2] for details. ■

Let us remark here that there is no condition on the conductors of π and Λ . One can relax the unitarity condition (see [2]) but these are satisfied for global applications. When L is a field extension, we have the following result.

1.2 Theorem. *Let L be a field. Assume that $c(\Lambda) \geq c(\pi) > 0$. Then $\text{Hom}_{T(F)}(\pi, \Lambda) \neq 0$, and for $0 \neq \ell \in \text{Hom}_{T(F)}(\pi, \Lambda)$ there exists a good test vector with*

$$K = \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{p}^{c(\Lambda)} \\ \mathfrak{p}^{c(\pi)-c(\Lambda)} & 1 + \mathfrak{p}^{c(\pi)} \end{bmatrix} \cap \text{GL}_2(F) = hK_{c(\pi)}h^{-1},$$

with

$$h = \begin{bmatrix} \varpi^{c(\Lambda)-c(\pi)} & \\ & 1 \end{bmatrix} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}.$$

Proof. If π is an irreducible principal series or a twist of the Steinberg representation by a ramified character, then we use the induced model for π . We define $\ell : V_\pi \rightarrow \mathbb{C}$ by the integral

$$\ell(f) := \int_{Z(F) \backslash T(F)} f(t) \Lambda^{-1}(t) dt.$$

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Here, $Z(F)$ is the center of G . Since $Z(F)\backslash T(F)$ is compact, this integral always converges. It can be shown that $\ell \neq 0$ for any $c(\pi)$ and $c(\Lambda)$ (see Section 4.1 of [2]). Let f_0 be the new form in the induced model. An explicit calculation shows that, if $c(\Lambda) \geq c(\pi) > 0$, then $\ell(\pi(h)f_0) \neq 0$, for h as in the statement of the theorem.

If π is the twist of the Steinberg representation by an unramified character, we can in fact do more than in the statement of the theorem. Let $\mathcal{B}(\pi, \Lambda)$ be the Λ -Waldspurger model for π consisting of functions on G transforming by Λ under left translation by $T(F)$. Let B_0 be the new form for π in $\mathcal{B}(\pi, \Lambda)$. Then B_0 is completely determined by its values on the representatives of the double cosets $T(F)\backslash G/K_1$. Using Hecke operators and Atkin-Lehner operators, we can find the explicit value of B_0 for all these representatives. The theorem follows.

If π is a supercuspidal representation, we use Mackey theory. We realize $\pi = c - \text{Ind}_J^G \rho$, where J is a compact (modulo center) subgroup of G and ρ a representation of J . Finding an appropriate ℓ amounts to understanding the intertwining between ρ and Λ . This boils down to looking at the double cosets for $T(F)\backslash G/J$ and showing that only one of them can support the intertwining : $T(F)hJ$, with h as in the statement of the theorem. Then, we show that the translate of the new form by h is indeed a test vector for ℓ . See [2] for details. ■

Let us remark here that the above calculations do not work for $c(\Lambda) < c(\pi)$. The crucial point is that, under the hypothesis $c(\Lambda) \geq c(\pi)$, certain multiplicative characters give rise to additive characters, which end up giving central values of epsilon factors which are non-zero.

2 Global applications

Let F be a number field and L a quadratic extension. Let π be a cuspidal, automorphic representation of $\text{GL}_2(\mathbb{A}_F)$ with trivial central character. Let Λ be an idele class character of \mathbb{A}_L^\times such that $\Lambda|_{\mathbb{A}_F^\times} = 1$. We are interested in the central value of the L -function $L(1/2, \pi_L \times \Lambda) = L(1/2, \pi, \theta_\Lambda)$. Here, π_L is the base change of π to $\text{GL}_2(\mathbb{A}_L)$ and θ_Λ is the theta function corresponding to Λ . There are several properties of these central values that are of interest – non-vanishing, positivity, sub-convexity, arithmetic properties. The main tool to study this central L -value is the relation it has to period integrals. Let us explain this.

Let D be a quaternion algebra over F containing L such that π has a Jacquet-Langlands transfer π' to $D^\times(\mathbb{A}_F)$. Note that we allow the possibility that D is the matrix algebra. As before, one can embed L^\times as a torus T in D^\times . For $\phi \in \pi'$, define the integral

$$P_D(\phi) = \int_{Z(\mathbb{A}_F)T(F)\backslash T(\mathbb{A}_F)} \phi(t)\Lambda^{-1}(t)dt.$$

In [10], using the theory of theta lifts, Waldspurger proved the beautiful formula

$$\frac{|P_D(\phi)|^2}{(\phi, \phi)} = \zeta(2) \prod_v \alpha_v(L, \Lambda, \phi) \frac{L(1/2, \pi_L \times \Lambda)}{L(1, \pi, \text{Ad})}.$$

Here, (\cdot, \cdot) is an inner product on π' . The terms $\alpha_v(L, \Lambda, \phi)$ are certain local integrals which are 1 for almost all places v and $L(s, \pi, \text{Ad})$ is the adjoint L -function of π . It turns out that, excepting for one unique choice of D , the period integrals $P_D(\phi)$ are trivially zero for local reasons. If one

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fixes this unique D , one gets the criteria for non-vanishing: $L(1/2, \pi_L \times \Lambda) \neq 0$ if and only if $P_D(\phi) \neq 0$.

In [5], Jacquet and Chen have studied the same L -value using a relative trace formula. They get a similar formula with the additional information that the local integrals are squares. This immediately leads to the positivity result $L(1/2, \pi_L \times \Lambda) \geq 0$. For any further information regarding the central L -values, one needs a more explicit formula, i.e., one needs to compute the local integrals for suitable choices of ϕ . For this, one needs to choose ϕ such that the local component of ϕ are precisely the good test vectors from the previous section. Using the Jacquet-Chen formula and the test vectors from the work of Gross and Prasad, explicit central value formulas were obtained by Martin and Whitehouse [6] assuming π and Λ have disjoint ramification.

In case of joint ramification, we obtain local test vectors from Theorems 1.1 and 1.2, which yields the desired global test vector ϕ . Let us now describe the L -value formula obtained by us more precisely.

Denote the absolute value of the discriminants of F and L by Δ and Δ_L . Let $e(L_v/F_v)$ be the ramification degree of L_v/F_v . Let S_{inert} be the set of places of F inert in L . Let $S(\pi)$ (resp. $S(\Lambda)$) be the set of finite places of F where π (resp. Λ) is ramified, $S_1(\pi)$ (resp. $S_2(\pi)$) the set of places where $c(\pi_v) = 1$ (resp. $c(\pi_v) \geq 2$), and $S_0(\pi) = S_2(\pi) \cup \{v \in S_1(\pi) : L_v/F_v \text{ is ramified and } \Lambda_v \text{ is unramified}\}$. Denote by $c(\Lambda)$ the absolute norm of the conductor of Λ .

2.1 Theorem. *Let π be a cuspidal automorphic representation of $\text{GL}_2(\mathbb{A}_F)$ with trivial central character and Λ a character of $\mathbb{A}_L^\times/L^\times \mathbb{A}_F^\times$. Assume $\epsilon(1/2, \pi_L \otimes \Lambda) = 1$. If $v < \infty$ is inert in L and $c(\pi_v), c(\Lambda_v) > 0$, then assume that $c(\Lambda_v) \geq c(\pi_v)$. Then, one can choose a test vector $\phi \in \pi'$ such that*

$$\begin{aligned} \frac{|P_D(\phi)|^2}{(\phi, \phi)} &= \frac{1}{2} \sqrt{\frac{\Delta}{c(\Lambda)\Delta_L}} L_{S(\Lambda)}(1, \eta) L_{S(\pi) \cup S(\Lambda)}(1, \eta) L_{S(\pi) \cap S(\Lambda)}(1, 1_F) L^{S(\pi)}(2, 1_F) \\ &\quad \times \prod_{v \in S(\pi) \cap S(\Lambda)^c} e(L_v/F_v) \prod_{v|\infty} C_v(L, \pi, \Lambda) \cdot \frac{L^{S_0(\pi)}(1/2, \pi_L \otimes \Lambda)}{L^{S_0(\pi)}(1, \pi, \text{Ad})}. \end{aligned}$$

Here (\cdot, \cdot) is the standard inner product on π' with respect to the measure on $D^\times(\mathbb{A}_F)$ which is the product of local Tamagawa measures. Also, C_v , for $v|\infty$ are explicit non-zero numbers obtained from the archimedean computation.

Proof. Let S be the set of all places of F including the archimedean ones and where any of L, π or Λ are ramified. For $v \in S_{\text{inert}}$, define

$$\tilde{J}_{\pi'_v}(f_v) = \int_{G'(F_v)} f_v(g) (\pi'_v(g) e'_v, e'_v) dg_v,$$

where e'_v is a norm 1 vector such that $\pi'_v(t) e'_v = \Lambda_v(t) e'_v$ for all $t \in T(F_v)$. For $v \in S - S_{\text{inert}}$, set

$$\tilde{J}_{\pi'_v}(f_v) = \sum_W \int_{F_v^\times} \pi'_v(f_v) W \left(\begin{bmatrix} a & \\ & 1 \end{bmatrix} \right) \Lambda \left(\begin{bmatrix} a & \\ & 1 \end{bmatrix} \right)^{-1} d^\times a \int_{F_v^\times} \overline{W \left(\begin{bmatrix} a & \\ & 1 \end{bmatrix} \right) \Lambda \left(\begin{bmatrix} a & \\ & 1 \end{bmatrix} \right)^{-1}} d^\times a,$$

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where $d^\times a$ is the local Tamagawa measure and W runs over an orthonormal basis for the local Whittaker model $\mathcal{W}(\pi_v, \psi_v)$. Let $f = \prod f_v \in C_c^\infty(G'(\mathbb{A}_F))$ with f_v the unit element of the Hecke algebra for $v \notin S$. Let $\pi'(f)$ be an orthogonal projection onto a 1-dimensional subspace $\langle \phi \rangle$. Then Jacquet-Chen prove the following identity in [5].

$$\frac{|P_D(\phi)|^2}{(\phi, \phi)} = \frac{1}{2} \prod_S \tilde{J}_{\pi'_v}(f_v) \prod_{v \in S_{\text{inert}}} 2\epsilon(1, \eta_v, \psi_v) L(0, \eta_v) \times \frac{L_S(1, \eta) L^S(1/2, \pi_L \otimes \Lambda)}{L^S(1, \pi, \text{Ad})}.$$

One can choose f so that it picks out the global test vector ϕ having local components as the good test vectors. If π_v or Λ_v is unramified or $v|\infty$, the integral $\tilde{J}_{\pi'_v}(f_v)$ has been computed in [6]. In the joint ramification case, we compute using the test vectors from the previous local section. If $c(\pi_v) \geq 2$, then

$$\tilde{J}_{\pi_v}(f_v) = q_v^{-c(\Lambda_v)} \frac{L(1, 1_{F_v}) L(1, \eta_v)}{e(L_v/F_v)}.$$

If $c(\pi_v) = 1$, then

$$\tilde{J}_{\pi_v}(f_v) = q_v^{-c(\Lambda_v)} \frac{L(1, 1_{F_v}) L(1, \eta_v)}{e(L_v/F_v) L(2, 1_{F_v})}.$$

Note that, in the above case, we can take $D_v^\times = \text{GL}_2(F_v)$ and hence, $\pi_v = \pi'_v$. Putting all the terms together, one gets the result of the theorem. \blacksquare

If $F = \mathbb{Q}$ and π corresponds to a holomorphic new form of square free level N with $N|c(\Lambda)$, then the above formula simplifies considerably:

2.2 Corollary. *Let f be a normalized holomorphic modular eigenform of weight k and square-free level N . Let S be the set of primes $p|N$ which split in L . Let Λ be any ideal class character of L such that $N|c(\Lambda)$ and $\epsilon(1/2, f \times \Lambda) = 1$. Then*

$$\frac{|P_D(\phi)|^2}{(\phi, \phi)} = \frac{C_\infty(L, f, \Lambda)}{2^{k+1} \sqrt{c(\Lambda)} \Delta_L} L_{S(\Lambda)}(1, \eta)^2 \prod_{p|N} (1 + p^{-1})^{\epsilon_p} \times \frac{L^S(1/2, f \times \Lambda)}{\langle f, f \rangle},$$

where ϵ_p is $+1$ if p splits in L and -1 otherwise, and $\langle \cdot, \cdot \rangle$ is the Petersson inner product.

Note that, in the above theorem and corollary, we still need to have additional information about the period to obtain properties of the central L -value. In certain special cases, one can have a better understanding of the periods. For example, in [4], the author considers a totally real field F and π corresponding to a new form f with parallel weight $(2, \dots, 2)$. Using the techniques of Cornut and Vatsal, and Theorem 2.1 above, the author obtains properties of $\text{ord}_\lambda(L^{\text{alg}}(1/2, f, \Lambda))$.

Another application of the explicit central value formula is to obtain an explicit average value formula for central L -values. As is clear from the theorem below, the formula for averages is much simpler, in particular, it does not involve the mysterious period anymore. Hence, non-vanishing results are immediate. Let us state the average value result next.

2.3 Theorem. *Let F be a totally real number field with $d = [F : \mathbb{Q}]$. Let $\mathcal{F}(\mathfrak{N}, 2\mathbf{k})$ be the set of cuspidal automorphic representations of $\text{GL}_2(\mathbb{A}_F)$ associated to the holomorphic Hilbert*

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modular eigen newforms of weight $2\mathbf{k}$ and level \mathfrak{N} , with $\mathbf{k} = (k_1, \dots, k_d) \neq (1, \dots, 1)$ and \mathfrak{N} squarefree. Let L be a totally imaginary quadratic extension of F , which is inert and unramified above each place $\mathfrak{p}|\mathfrak{N}$. Fix a unitary character Λ of $\mathbb{A}_L^\times/L^\times\mathbb{A}_F^\times$, and let \mathfrak{C} be the norm of its conductor in F . Suppose $\mathfrak{N} = \mathfrak{N}_0\mathfrak{N}_1$ and $\mathfrak{C} = \mathfrak{C}_0\mathfrak{N}_1$ with \mathfrak{N}_0 , \mathfrak{N}_1 and \mathfrak{C}_0 all coprime. Assume \mathfrak{N}_1 is odd, and that the number of primes dividing \mathfrak{N}_0 has the same parity as d . Further assume that for each infinite place v of F , $k_v > |m_v|$ where $\Lambda_v(z) = (z/\bar{z})^{m_v}$.

Then, if $|\mathfrak{N}_0| > d_{L/F}(|\mathfrak{C}_0|/|\mathfrak{N}_1|)^{h_F}$, where h_F is the class number of F , we have

$$\prod_{v|\infty} \binom{2k_v - 2}{k_v - m_v - 1} \sum_{\mathfrak{N}'} \sum_{\pi \in \mathcal{F}(\mathfrak{N}', 2\mathbf{k})} \frac{L(1/2, \pi_L \otimes \Lambda)}{L^{S(\mathfrak{N})}(1, \pi, Ad)} \\ = 2^{2-d} \Delta^{3/2} |\mathfrak{N}| L_{S(\mathfrak{N}_0)}(2, 1_F) L_{S(\mathfrak{N}_1)}(1, 1_F) L^{S(\mathfrak{C}_0)}(1, \eta),$$

where \mathfrak{N}' runs over ideals dividing \mathfrak{N} which are divisible by \mathfrak{N}_0 , and $S(\mathfrak{J})$ denotes the set of all primes dividing \mathfrak{J} .

The theorem is proven by computing the geometric sides of a trace formula. The Theorem specializes to [1, Thm 1.1] in the case that \mathfrak{N} and \mathfrak{C} are coprime, i.e., $\mathfrak{N} = \mathfrak{N}_0$. One can use the above formula together with formulas for smaller levels to get both explicit bounds and asymptotics for average values over just the forms of exact level \mathfrak{N} . We do this in the case \mathfrak{N}_1 is prime. This immediately implies $L(1/2, \pi_L \otimes \Lambda) \neq 0$ for some $\pi_L \in \mathcal{F}(\mathfrak{N}, 2\mathbf{k})$.

Lastly, we include another application of Theorem 2.3 when $\mathfrak{N} = \mathfrak{N}_0$. Here, having an exact formula for the average value over newforms allows us to deduce the nonvanishing mod p of the algebraic part $L^{\text{alg}}(1/2, \pi_L \otimes \Lambda)$ of the central value for p suitably large.

2.4 Theorem. *With notation and assumptions as in Theorem 2.3, suppose $|\mathfrak{N}| > d_{L/F}|\mathfrak{C}|^{h_F}$, \mathfrak{N} is coprime to \mathfrak{C} , and, for each $v|\infty$, m_v is even. Let p be an odd rational prime satisfying $p > q + 1$ for all primes $q \in S(\Omega)$, and \mathcal{P} a prime of \mathbb{Q} above p . Then there exists $\pi \in \mathcal{F}(\mathfrak{N}, 2\mathbf{k})$ such that*

$$L^{\text{alg}}(1/2, \pi_L \otimes \Lambda) \not\equiv 0 \pmod{\mathcal{P}}.$$

This generalizes a theorem of Michel and Ramakrishnan [7] on the case $F = \mathbb{Q}$ and $\mathfrak{N} = N$ is prime. The parity condition on m_v ensures that Λ is algebraic and that the above central value is critical.

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